Optimal control system of a microbeam model with the boundary term

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Abstract

In this paper, a hyperbolic system governed by boundary control, modeled as an optimal control problem is discussed. The control problem is formulated using boundary control mechanisms for the microbeam model to control undesirable free vibrations in the system. Wellposedness of the optimal solution on the control set is demonstrated and controllability of the problem is investigated. Solution procedure of the boundary control characterization of the microbeam model is examined by Maximum Principle. The necessary conditions for the optimal control problem are obtained thanks to this principle and these conditions are shown to be also sufficient conditions due to convexity. The proposed approach is based on transforming the problem into a system of partial differential equations. The obtained distributed parameter system model includes state and costate variables with terminal time initial conditions. An eigenfunction expansion method is used for the solution of the optimality conditions derived from the Maximum principle. Numerical results are obtained by using the computer codes produced in MATLAB© and presented in graphical and table forms. Numerical simulation studies show the applicability and effectiveness of this approach.

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1 Introduction

Microbeams are beams of typically on the order of microns and sub-microns thicknesses that are common in micro-and nano-electro-mechanical (MEMS and NEMS) areas. Micro-electronics, micro-actuators and micro-sensors are the parts that make up the system called MEMS [1, 2, 3, 4]. With the rapid development of nanotechnology in recent years, micro-electro-mechanical systems are entering our lives more intensively [5, 6]. Microbeams can be composed of different material combinations like metals, conventional silicone-based materials, polymers or functionally graded materials [7, 8, 9, 10]. The control design of the microbeam by using control actuators is an important research area [11, 12, 13]. Control mechanisms can be applied by means of boundary conditions on the system or through an internal force. Boundary control is a way to control of a distributed parameter system in which the control action is implemented to the system via its boundary conditions.

Guzman and Zhu [14] used a single boundary control to examine the exact controllability property of a microbeam. Korpeoglu et al. [15] studied optimal boundary control for a beam modeled based on second strain gradient theory by means of maximum principle to control undesirable vibrations in the system. Zhao et al. [16] developed the nonlinear microbeam model by using the strain gradient theory and also the nonlinear free vibration is investigated. Kong et al. [17] used

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a variational approach and the strain gradient elasticity theory to derive the static and dynamic models for Euler–Bernoulli beams. The second strain gradient theory is very strong non-classical continuum theory that captures the behavior of micrometer and nanometer sized structures.

In this paper, the optimal control of a microbeam model is designed by means of controls placed on the boundary condition with its 4th order space derivative. The performance index seeks to minimize the magnitude of the performance measure that is defined as a dynamic response the beam as well as the control input over the time interval. Maximum principle is applied to get the optimal control solutions. The costate (adjoint) variable is defined to reformulate the optimal control problem in terms of Hamiltonian function. The state equations, the costate equations, optimality conditions and terminal time initial conditions are expressed with the help of Hamiltonian. Simulation operation is performed to show the effect of the suggested controller on the microbeam.

Numerical results are presented using computer simulation produced in MATLAB to confirm that the control scheme is successful in producing the desired result and appropriate to reduce the vibration of the beam. Besides, the mathematical methods suggested in this article are precise and applicable. The importance of the study is also that it deals with optimal boundary control that is achieved based on space derivatives of boundary states of a microbeam model by means of Pontryagin's maximum principle.

2 Mathematical modeling

The equation of motion for the microbeam model derived based on the strain gradient theory and Hamilton's principle, governed by boundary controls $p(t)$ is given by [16],

$$\begin{cases} \rho A z_{tt} + S z_{xxxx} - K z_{xxxxxx} = f(x,t), \qquad (x,t) \in (0,L)\times(0,T), \\ z(0,t) = 0, \qquad z(L,t) = 0, \\ z_{xx}(0,t) = 0, \qquad z_{xx}(L,t) = 0, \\ z_{xxxx}(0,t) = p(t), \qquad z_{xxxx}(L,t) = p(t), \\ z(x,0) = z_a(x), \qquad z_t(x,0) = z_b(x). \end{cases} \tag{2.1}$$

where $z = z(x,t) \in D = (0,L)\times(0,T)$ is the deflection of the microbeam, $A > 0$, $L > 0$ and $\rho > 0$ are the parameters correspond the cross-sectional area, the length and the density of the microbeam, respectively. The parameters S and K are size effects of the microbeam in the order of microns [22]

$$S = EI + \mu A(2\ell_0^2 + \frac{43}{225}\ell_1^2 + \ell_2^2), \qquad K = \mu A(2\ell_0^2 + \frac{4}{5}\ell_1^2) \tag{2.2}$$

where $E > 0$ is the Young modulus, $\mu > 0$ the shear modulus and $I > 0$ the area moment of inertia. $\ell_0 > 0$, $\ell_1 > 0$ and $\ell_2 > 0$ are the independent material parameters. The Euler-Bernoulli beam model is obtained when $\ell_0 = \ell_1 = \ell_2 = 0$. In the next section, the wellposedness results and controllability property are presented.

3 Wellposedness and controllability

The aim of this section is to show the wellposedness and controllability of the control system (2.1). In consequence of being able to say the existence of the solution using Picard's existence theorem [19], it is required to have solutions for the equation system (2.1) with data $p(t) \in L^2(0,T), f \in L^2(0,T;L^2(0,L)), z_a(x) \in H^1(0,L), z_b(x) \in L^2(0,L)$, $z, \frac{\partial^i z}{\partial t^i}, \frac{\partial^i z}{\partial x^i}, \in L^2(\mathcal{D}), \quad i = 0,1,...,6.$ $L^2(\mathcal{D})$

denote the class of square integrable functions with a usual inner product and norm in the domain D. Moreover, equation system (2.1) can be written as ordinary differential equation form and have a solution under favour of linear Picard-Lindelöf existence-uniqueness theorem. The following lemma shows the uniqueness of the solution to Eqs. (2.1) based on energy method.

Lemma 3.1. Equation system (2.1) with $f = 0$ and $p = 0$ has a unique solution.

Proof. Suppose that the problem has two solution $z_1(x,t) \neq z_2(x,t)$ with $f = 0, p = 0$ and $z_a(x) \in H^1(0,L), z_b(x) \in L^2(0,L)$. Set the difference function $w(x,t) = z_1(x,t) - z_2(x,t)$ for the microbeam

$$\rho A w_{tt} + S w_{xxxx} - K w_{xxxxxx} = 0, \qquad 0 \leq t \leq T, \qquad 0 \leq x \leq L, \tag{3.1}$$

with zero initial conditions

$$w(x,0) = w_t(x,0) = 0, \tag{3.2}$$

and the following boundary conditions

$$w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = w_{xxxx}(0,t) = w_{xxxx}(L,t) = 0. \tag{3.3}$$

If $w(x,t)$ is shown that it is identically zero in D, the uniqueness of the solution is obtained. Let us examine the energy integral [14] as

$$E(t) := \frac{1}{2}\int_0^L \big(\mid w_t(x,t) \mid^2 + \frac{S}{\rho A} \mid w_{xx}(x,t) \mid^2 + \frac{K}{\rho A} \mid w_{xxx}(x,t) \mid^2 \big) dx, \qquad t \in [0,T]. \tag{3.4}$$

Differentiating $E(t)$ with respect to t gives

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_0^L \{ w_t w_{tt} + \frac{S}{\rho A} w_{xx} w_{xxt} + \frac{K}{\rho A} w_{xxx} w_{xxxt} \} dx \\ &= \int_0^L \{ w_{tt} + \frac{S}{\rho A} w_{xxxx} + \frac{K}{\rho A} w_{xxxxxx} \} w_t(x,t) dx \\ &+ \left\{ \frac{S}{\rho A}(w_{xx} w_{xt} - w_{xxx} w_t)\Big|_0^\ell + \frac{K}{\rho A}(w_{xxx} w_{xxt} - w_{xxxx} w_{xt} + w_{xxxxx} w_t)\Big|_0^\ell \right\}. \end{aligned} \tag{3.5}$$

By using Eq. (3.1) and boundary conditions (3.3), it follows that $\frac{dE(T)}{dt} = 0$, that is, $E(t) = constant$.
Taking the initial conditions (3.2) into consideration, the following equality holds

$$\begin{aligned} E(T) &= constant = E(0) \\ &= \frac{1}{2}\int_0^L \left\{ w_t^2(x,t) + \frac{S}{\rho A} w_{xx}^2(x,t) + \frac{K}{\rho A} w_{xxx}^2(x,t) \right\}\Big|_{t=0} dx = 0. \end{aligned}$$

Then it follows from Eq. (3.4) and from the initial conditions (3.2) that $w(x,t)$ is identically equal to zero in $\mathcal{D}$, that is, $z_1 = z_2$, which completes the proof. Q.E.D.

By considering the uniqueness solution of the beam system, it is determined that the control function is unique by the reason of the uniqueness of $p(t)$. In this case, the studied system is observable due to having a unique solution and unique control function. Briefly, the system defined by Eqs. (2.1) is controllable according to the Hilbert uniqueness method [20, 21].

4 Optimal control design

It is desired to determine an optimal control function $p(t)$ that is placed on the boundary conditions for damping out the undesired vibrations. In accordance with this purpose the performance index that is minimized over the time interval $0 \leq t \leq T$ is defined in two parts: The first term measures the dynamical response of the system at the terminal time T and the second term is the penalty function that minimizes control force's expenditure of used in $[0, T]$. A control that satisfies the control constraints during the time interval $[0, T]$ is called an admissible control and the set of admissible controls by P_{ad} is as given follows:

$$P_{ad} = \{p(t)|p \in L^2(0,T), \quad |p(t)| \leq a_0 < \infty, \quad a_0 \text{ is a constant } \} \tag{4.1}$$

The performance index is given by

$$\mathcal{J}[p(t)] = \int_0^L [\mu_1 z^2(x,T) + \mu_2 z_t^2(x,T)]dx + \int_0^T \mu_3 p(t)^2 dt, \tag{4.2}$$

where μ_1, μ_2 and μ_3 are weight coefficients satisfying $\mu_1, \mu_2 \geq 0$, $\mu_3 > 0$ and $\mu_1 + \mu_2 \neq 0$, . The performance index is selected as a sum of two integrals. The first integral is the dynamic response of the beam and seeks to minimize the vibrations at the terminal time $t = T$. The second integral is the penalty function that minimizes the magnitude of the control over the range $0 \leq t \leq T$. The optimal control problem is specified as

$$\mathcal{J}(p^o(t)) = \min_{p(t) \in P_{ad}} \mathcal{J}(p(t)) \tag{4.3}$$

subject to the system (2.1).

5 Boundary control

To apply maximum principle, let us define the adjoint system with the adjoint (costate) variable v corresponding to (2.1) as follows:

$$\rho A v_{tt} + S v_{xxxx} - K v_{xxxxxx} = 0 \tag{5.1}$$

with homogeneous boundary conditions

$$\begin{aligned} v(0,t) &= v(L,t) = 0, \\ v_{xx}(0,t) &= v_{xx}(L,t) = 0, \\ v_{xxxx}(0,t) &= v_{xxxx}(L,t) = 0, \end{aligned} \tag{5.2}$$

and terminal conditions

$$\begin{aligned} \rho A v_t(x,T) &= 2\mu_1 z(x,T) \\ -\rho A v(x,T) &= 2\mu_2 z_t(x,T). \end{aligned} \tag{5.3}$$

Optimal control problem is reformulated using Pontryagin's maximum principle, which asserts that a necessary condition for optimal control function that minimizes the Pontryagin's Hamiltonian.

Since the Hamiltonian satisfies $\dfrac{\partial \mathcal{H}}{\partial p}(t, z^o(t), p^o(t), v^o(t)) = 0$ and $\dfrac{\partial^2 \mathcal{H}}{\partial p^2} > 0$, the Pontryagin's principle is also sufficient to be an optimal solution. It can be also said that the optimality conditions obtained from results of maximum principle are sufficient conditions because of the convexity property of the performance index. Pontryagin's principle gives a clear statement for the optimal control function by relating the state variable and the optimal control function implicitly. In this context, Pontryagin's principle is applicable to the optimal control problem (4.3) as described in the next section.

6 Derivation of the maximum principle

The Pontryagin's maximum principle for the optimal control problem is given as follows:

Theorem 6.1. If the optimal control function $p^o(t) \in P_{ad}$, which causes the system (2.1) minimizes the Hamiltonian so that

$$\mathcal{H}(t; v, p) = KR(t)p(t) + \mu_3 p^2(t) \tag{6.1}$$

where

$$R(t) = v_x(L,t) - v_x(0,t) \tag{6.2}$$

then

$$\mathcal{J}(p^o) \le \mathcal{J}(p). \tag{6.3}$$

Proof. Let us form an operator

$$\Gamma(z) = \rho A z_{tt} + S z_{xxxx} - K z_{xxxxxx} \tag{6.4}$$

and differences

$$\begin{aligned} \Delta z &= z(x,t) - z^o(x,t), & (6.5)\\ \Delta p &= p(t) - p^o(t). & (6.6) \end{aligned}$$

Evaluating the operator and differences gives

$$\Gamma(\Delta z) = 0, \tag{6.7}$$

with the boundary conditions

$$\begin{gathered} \Delta z(0,t) = \Delta z(L,t) = 0,\\ \Delta z_{xx}(0,t) = \Delta z_{xx}(L,t) = 0,\\ \Delta z_{xxxx}(0,t) = \Delta z_{xxxx}(L,t) = \Delta p(t) \end{gathered} \tag{6.8}$$

and initial conditions

$$\Delta z(x,0) = \Delta z_t(x,0) = 0. \tag{6.9}$$

The following relation yields

$$\begin{aligned} \int_0^L \int_0^T (\Delta z \Gamma(v) - v\Gamma(\Delta z)) dt dx = \int_0^L \int_0^T \{ & \underbrace{[\rho A(\Delta z v_{tt} - v\Delta z_{tt})]}_{I} + \underbrace{S(\Delta z v_{xxxx} - v\Delta z_{xxxx})}_{II} \\ & - \underbrace{K(\Delta z v_{xxxxxx} - v\Delta z_{xxxxxx})}_{III} \} dt dx = 0 \end{aligned} \tag{6.10}$$

Carrying out integration by parts for I, II, III and using the terminal conditions Eqs. (5.3) gives

$$I = \int_0^L \int_0^T \rho A(\Delta z v_{tt} - v\Delta z_{tt})dtdx = \rho A \int_0^L (\Delta z(x,T)v_t(x,T) - \Delta z_t(x,T)v(x,T))dx, \quad (6.11)$$

$$\begin{aligned} II = \int_0^L \int_0^T S(\Delta z v_{xxxx} - v\Delta z_{xxxx})dtdx = S\int_0^T \{&\Delta z(L,t)v_{xxx}(L,t) - \Delta z(0,t)v_{xxx}(0,t) \\ &- \Delta z_x(L,t)v_{xx}(L,t) + \Delta z_x(0,t)v_{xx}(0,t) \\ &+ \Delta z_{xx}(L,t)v_x(L,t) - \Delta z_{xx}(0,t)v_x(0,t) \\ &- \Delta z_{xxx}(L,t)v(L,t) + \Delta z_{xxx}(0,t)v(0,t)\}dt, \end{aligned} \quad (6.12)$$

$$\begin{aligned} III = \int_0^L \int_0^T -K(\Delta z v_{xxxxxx} - v\Delta z_{xxxxxx})dtdx = -K\int_0^T &\{\Delta z(L,t)v_{xxxxx}(L,t) - \Delta z(0,t)v_{xxxxx}(0,t) \\ &- \Delta z_x(L,t)v_{xxxx}(L,t) + \Delta z_x(0,t)v_{xxxx}(0,t) \\ &+ \Delta z_{xx}(L,t)v_{xxx}(L,t) - \Delta z_{xx}(0,t)v_{xxx}(0,t) \\ &- \Delta z_{xxx}(L,t)v_{xx}(L,t) + \Delta z_{xxx}(0,t)v_{xx}(0,t) \\ &+ \Delta z_{xxxx}(L,t)v_x(L,t) - \Delta z_{xxxx}(0,t)v_x(0,t)\}dt. \end{aligned} \quad (6.13)$$

Combining these three results yields

$$\begin{aligned} \int_0^L \int_0^T (\Delta z\Gamma(v) - v\Gamma(\Delta z))dtdx &= \rho A \int_0^L (\Delta z(x,T)v_t(x,T) - \Delta z_t(x,T)v(x,T))dx \\ &- K\int_0^T \{\Delta z_{xxxx}(L,t)v_x(L,t) - \Delta z_{xxxx}(0,t)v_x(0,t)\}dt \\ &= 0. \end{aligned} \quad (6.14)$$

Bringing into focus on the deviations of performance index gives

$$\begin{aligned} \Delta\mathcal{J}(p) &= \mathcal{J}(p) - \mathcal{J}(p^o) \\ &= \int_0^L \{\mu_1[z^2(x,T) - z^{o^2}(x,T)] + \mu_2[z_t^2(x,T) - z_t^{o^2}(x,T)]\}dx, \\ &+ \mu_3 \int_0^T [p^2(t) - p^{o^2}(t)]dt. \end{aligned} \quad (6.15)$$

The values of $z^2(x,T)$ and $z_t^2(x,T)$ around $z^{o^2}(x,T)$ and $z_t^{o^2}(x,T)$ by Taylor Series expansion, respectively, are

$$\begin{aligned} z^2(x,T) - z^{o^2}(x,T) &= 2z^o(x,t)\Delta z(x,T) + r_1 \\ z_t^2(x,T) - z_t^{o^2}(x,T) &= 2z_t^o(x,t)\Delta z_t(x,T) + r_2 \end{aligned} \quad (6.16)$$

in which

$$r_1 = 2(\Delta z)^2 + ... > 0, \quad r_2 = 2(\Delta z_t)^2 + ... > 0. \quad (6.17)$$

Substituting Eqs. (6.16) into Eq. (6.15) yields

$$\Delta\mathcal{J}(p)=\int_0^L\{2\mu_1[z^o(x,T)\Delta z^o(x,T)+r_1]+2\mu_2[z_t^o(x,T)\Delta z_t^o(x,T)+r_2]\}dx+\mu_3\int_0^T[p^2(t)-p^{o^2}(t)]dt. \tag{6.18}$$

Applying the fact $2\mu_1 r_1+2\mu_2 r_2\geq 0$ gives

$$\int_0^T K\{v_x(L,t)-v_x(0,t)\}\Delta p(t)dt+\mu_3\int_0^T\{p^2(t)-p^{o^2}(t)\}dt\geq 0. \tag{6.19}$$

Then Pontryagin's Hamiltonian is of the form

$$\mathcal{H}(t;v,p)=KR(t)p(t)+\mu_3 p^2(t), \tag{6.20}$$

where $R(t)=v_x(L,t)-v_x(0,t)$. Thus,

$$\min\mathcal{H}(t;v,p)=\mathcal{H}(t;v^o,p^o) \tag{6.21}$$

yields $\mathcal{J}(p^o)\leq\mathcal{J}(p)$. Q.E.D.

7 Simulations

In this section, the simulation operation with regard to Equation system (2.1) is performed with boundary conditions. The efficiency and competence of the boundary control algorithm introduced are simulated through computer codes produced in MATLAB . Optimal solutions of the microbeam for the case with terminal time $T=4$s and weight coefficient $\mu_3=10^{-3}$ are shown in the simulations. The other weight coefficients μ_1 and μ_2 in the first integral of the performance index functional are considered as $\mu_1=\mu_2=1$. It is known that the material length scale parameter of a microbeam has been experimentally obtained as $L=17.6\mu$m by Lam. [22], so the length of the microbeam is taken as $L=17.6\mu$m . The component part of the microbeam used in this study include aluminum $E=70\quad$ GMpa, $\rho=2720$kg/m^3, $v=0.3$ [23] . For the purpose of the simplicity, all three material length scale parameters are assumed to be the same, i.e., $\ell_0=\ell_1=\ell_2=L$ within the microscale beam model. The values of the velocity and displacement of the beam are calculated at the exact middle point. The introduced control algorithm is effective even if the coefficients are chosen as desired. The response of the microbeam is analyzed subject to the initial conditions

$$z(x,0)=\sqrt{2}\sin(\pi x),\qquad z_t(x,0)=0. \tag{7.1}$$

The controlled/uncontrolled displacements are shown in Fig. 1. It is observed that the vibrations are close to zero at the terminal time. The controlled/uncontrolled velocities are also given in Fig. 2. As it is clear with the help of the figures, after applying the control action, vibrations of the system are suppressed using minimum level of control.

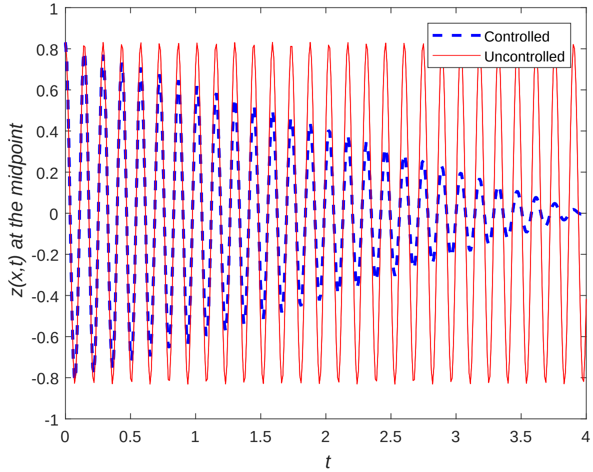

FIGURE 1. Controlled and uncontrolled displacements.

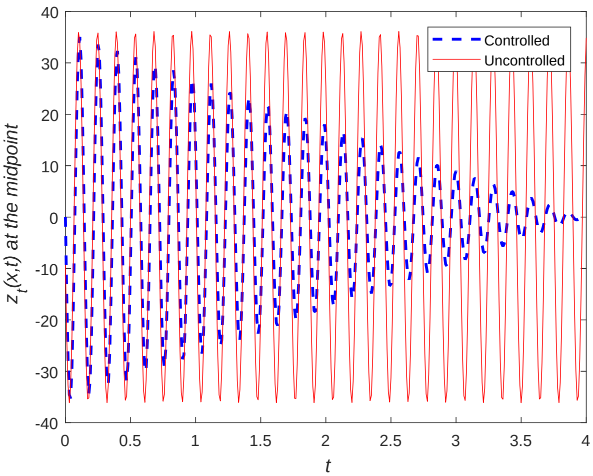

FIGURE 2. Controlled and uncontrolled velocities.

8 Conclusion

In this paper, optimal control solutions to a hyperbolic equation with boundary controls are obtained. The examined model is used to describe nonlinear microbeam models based on the strain gradient theory and the Hamilton's principle. The performance index to be minimized indicates controlling the dynamic response of the system while an affordable control is in use. Vibrations of the microbeam is suppressed by designing proper boundary control mechanisms. This control mechanism is achieved with the forth order space derivatives of boundary states of the beam. Maximum principle and eigenfunction expansion method as a numerical technique are used in the presented approach. Besides, wellposedness of the optimal solution on the control set is presented and controllability of the problem is analyzed. Moreover, to demonstrate the performance of the designed boundary controllers via numerical simulation, the computer codes produced in MATLAB© are used. Finally, the theoretical results obtained in the study are verified with the help of numerical simulations.

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